Hyperbolic geometry, continued fractions and classification of the finitely generated totally ordered dimension groups

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Abstract

We classify polycyclic totally ordered dimension groups, i.e. dimension groups generated by dense embeddings of the lattice $\mathbb{Z}^n$ in the real line $\mathbb{R}$. Our method is based on geometry of simple geodesics on the modular surface of genus $g \geq 2$. The main theorem says that isomorphism classes of the polycyclic totally ordered dimension group are bijective with the reals α modulo the action of the group $GL(2, \mathbb{Z})$. The result is an extension of the Effros-Shen classification of the dicyclic dimension groups.

Key words and phrases: dimension group, geodesic lamination, continued fraction

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1 Introduction

The dimension of a subspace of the euclidean space $\mathbb{R}^n$ can be 1, 2, 3 or more. It is no longer true that dimension of any non-trivial subspace of the euclidean space $\mathbb{R}^\infty$ is a positive integer. It was discovered by von Neumann that there exist linear subspaces in $\mathbb{R}^\infty$ of a non-integer (continuous) dimension. In fact, there exists a dimension function on projections in a von Neumann algebra, which takes on any real value in the interval $[0, 1]$.

Unlike von Neumann algebras, the dimension function on the projections of a C^*-algebra, A, takes value in an abelian group, $K_0(A)$, rather than in $\mathbb{R}$ [5], Ch. 1. The range of dimension function in $K_0(A)$ is known as a *dimension group* of A. The classification of such groups is a difficult open problem.

In [6] Effros and Shen classified the dicyclic dimension groups, i.e. dimension groups inside the abelian group $\mathbb{Z}^2$. The result says that each dicyclic group can be assigned a positive irrational number α, defined up to the action of matrix group $GL(2, \mathbb{Z})$, and such that if

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$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}} = [a_0, a_1, a_2, \dots], \tag{1}$$

is a regular continued fraction of α, then one gets a representation of the dicyclic group via simplicial dimension groups:

$$\mathbb{Z}^2 \overset{\left(\begin{smallmatrix} 0 & 1 \\ 1 & a_0 \end{smallmatrix}\right)}{\longrightarrow} \mathbb{Z}^2 \overset{\left(\begin{smallmatrix} 0 & 1 \\ 1 & a_1 \end{smallmatrix}\right)}{\longrightarrow} \mathbb{Z}^2 \overset{\left(\begin{smallmatrix} 0 & 1 \\ 1 & a_2 \end{smallmatrix}\right)}{\longrightarrow} \dots \tag{2}$$

see Theorem 3.2 of [6]. The irrational α is a *slope* of the straight line in the plane, which is the universal cover of a two-dimensional torus, see [6] Theorem 2.1. The range of dimension function (an order) in $\mathbb{Z}^2$ is described by the cone $P_\alpha = \{(k,l) \in \mathbb{Z}^2 \mid \alpha k + l \geq 0\}$, and it is shown that every dicyclic dimension group arises in this way.

An objective of present note is similar classification of the polycyclic dimension groups, i.e. the dimension groups inside the abelian group $\mathbb{Z}^n$. Recall that a universal cover for the surfaces of genus $g \geq 2$ is no longer the euclidean plane but a hyperbolic plane, $\mathbb{H}$. It is well known that a substitution for the straight lines in $\mathbb{H}$ is given by the simple geodesics, i.e. geodesics with no self-crossing or self-tangent points. Thus, to classify the polycyclic dimension groups, one needs:

(i) to define a dimension group, G, coming from the simple geodesic, γ, on a Riemann surface of genus $g \geq 2$;

(ii) to introduce a slope (a real number α) of γ on the surface;

(iii) to construct a simplicial approximation of G in terms of the given slope α.

The realization of (i) – (iii) is as follows. The closure, $\bar{\gamma}$, of a simple non-periodic geodesic γ consists of a continuum of disjoint non-periodic geodesics known as a *geodesic lamination* λ [4]. Let $|\lambda|$ be the total number of the principal regions of λ (*ibid.*, p.60) and $n = 2g + |\lambda| - 1$, where g is the genus of Riemann surface carrying the lamination λ. It is well known that the set of invariant transversal measures of λ is a convex compact set, Δ_{k-1}, of dimension $1 \leq k \leq \left[\frac{n}{2}\right]$, where $[\bullet]$ is the integer part of a number [13]. By G_λ we understand a dimension group inside the abelian group $\mathbb{Z}^n$, whose state space $S(G_\lambda)$ is isomorphic to Δ_{k-1}. The assignment of G_λ to λ is unique [9]. To implement (ii), one needs to restrict to the Legendre laminations on a modular surface $X(N) = \mathbb{H}/\Gamma(N)$, where $\Gamma(N)$ is the principal congruence group. The slope α of λ is given by a regular continued fraction generated by the periodic geodesics, which approximate λ in the space of geodesic laminations on $X(N)$. Finally, to realize item (iii), one needs to restrict to a generic case $k = 1$ (i.e. totally ordered dimension groups). The condition secures the convergence of a Jacobi-Perron continued fraction attached to lamination λ_α and one can apply the known theorem of Effros and Shen [7] on the unimodular dimension groups.

To formulate our main results, let $\mathcal{F}_\alpha$ be a foliation on the surface $X = X(N)$ obtained by a blow-down of the lamination λ_α (see appendix). For simplicity, we let $\mathcal{F}_\alpha$ be given by trajectories of a closed one-form, ω_α, on X. The periods of ω_α in a basis $\{\gamma_1, \dots, \gamma_n\}$ in the group $H_1(X, Sing\ \omega_\alpha; \mathbb{Z})$ will be denoted by $\lambda_i = \int_{\gamma_i} \omega_\alpha$. Let $\theta_i = \lambda_{i-1}/\lambda_1$ and $\theta = (\theta_1, \dots, \theta_{n-1})$. Finally, let the Jacobi-Perron continued fraction of θ be:

$$\begin{pmatrix} 1 \\ \theta \end{pmatrix} = \lim_{k\to\infty} \begin{pmatrix} 0 & 1 \\ I & b_0 \end{pmatrix} \dots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix} \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix}, \tag{3}$$

where I is the unit matrix, $b_i = (b_1^{(i)}, \dots, b_n^{(i)})^T$ a vector of the non-negative integers and $\mathbb{I} = (0, \dots, 0, 1)^T$. A summary of our results can be formulated as follows.

Theorem 1 *Each finitely generated totally ordered dimension group G can be indexed by a positive irrational number $\alpha \in U$, where U is a generic subset of $\mathbb{R}$. The parametrization, G_α, has the following properties:*

(i) G_α and $G_{\alpha'}$ are order-isomorphic if and only if $\alpha' = \alpha \mod GL(2,\mathbb{Z})$;

(ii) G_α is the limit of the following simplicial dimension groups:

$$\mathbb{Z}^n \stackrel{\begin{pmatrix} 0 & 1 \\ I & b_0 \end{pmatrix}}{\longrightarrow} \mathbb{Z}^n \stackrel{\begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix}}{\longrightarrow} \mathbb{Z}^n \dots \tag{4}$$

The article is organized as follows. In section 2 we recall some useful definitions and facts on the dimension groups. The dimension group G_λ is introduced in section 3. The Legendre laminations are revised in section 4. Theorem 1 is proved in section 5. Finally, in section 6 a brief account of geodesic laminations, modular surfaces and the Jacobi-Perron fractions is given.

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2 Dimension groups

We use $\mathbb{Z}, \mathbb{Z}^+, \mathbb{Q}$ and $\mathbb{R}$ for the integers, positive integers, rationals and reals, respectively and $GL(n,\mathbb{Z})$ for the group of $n \times n$ matrices with entries in $\mathbb{Z}$ and determinant ± 1.

By an *ordered group* we shall mean an abelian group G together with a subset $P = G^+$ such that $P + P \subseteq P, P \cap (-P) = \{0\}$, and $P - P = G$. We call P the *positive cone* on G. We write $a \le b$ (or $a < b$) if $b - a \in P$ (or $b - a \in P \backslash \{0\}$).

G is said to be a *Riesz group* if:

(i) $g \in G$ and $ng \ge 0, n \in \mathbb{Z}^+$ implies $g \ge 0$;

(ii) $u, v \le x, y$ in G implies existence of $w \in G$ such that $u, v \le w \le x, y$.

Given ordered groups G and H, we say that a homomorphism $\varphi : G \to H$ is *positive* if $\varphi(G^+) \subseteq H^+$, and that $\varphi : G \to H$ is an *order isomorphism* if $\varphi(G^+) = H^+$.

A positive homomorphism $f : G \to \mathbb{R}$ is called a *state* if $f(u) = 1$, where $u \in G^+$ is an order unit of G. We let $S(G)$ be the *state space* of G, i.e. the set of states on G endowed with natural topology [5].

$S(G)$ is a compact convex subset of the vector space $Hom\ (G, \mathbb{R})$. By the Krein-Milman theorem, $S(G)$ is the closed convex hull of its extreme points, which are called *pure states*.

An ordered abelian group is a *dimension group* if it is order isomorphic to $\lim_{m.n\to\infty}(\mathbb{Z}^{r(m)}, \varphi_{mn})$, where the $\mathbb{Z}^{r(m)}$ are simplicially ordered groups (i.e. $(\mathbb{Z}^{r(m)})^+ \cong \mathbb{Z}^+ \oplus \ldots \oplus \mathbb{Z}^+$), and the φ_{mn} are positive homomorphisms. The dimension group G is said to be *unimodular* if $r(m) = Const = r$ and φ_{mn} are positive isomorphisms of $\mathbb{Z}^r$. In other words, G is the limit

$$\mathbb{Z}^r \xrightarrow{\varphi_0} \mathbb{Z}^r \xrightarrow{\varphi_1} \mathbb{Z}^r \xrightarrow{\varphi_2} \ldots, \tag{5}$$

of matrices $\varphi_k \in GL(r, \mathbb{Z}^+)$.

The Riesz groups are dimension groups and vice versa. The Riesz groups can be viewed as the abstract dimension groups, while dimension groups as a representation of the Riesz groups by the infinite sequences of positive homomorphisms.

3 Dimension groups generated by simple geodesics

Let γ be a simple non-periodic geodesic on a Riemann surface X of genus $g \geq 2$. The closure $\bar{\gamma}$ contains a continuum of non-periodic pairwise disjoint simple geodeisics, which form a perfect (Cantor) set on the surface X. The $\bar{\gamma}$ is a geodesic lamination, which we shall denote by λ. The number of principal regions of λ will be denoted by $|\lambda|$.

It is well known that λ corresponds to a foliation, $\mathcal{F}_\lambda$, on X. The $\mathcal{F}_\lambda$ is a foliation with $|\lambda|$ singular points of the saddle type. The $\mathcal{F}_\lambda$ is obtained from λ by a blow-down homotopy [14]. Denote by Φ_X the space of foliations on X, whose singularity set coincides with such of $\mathcal{F}_\lambda$. The coordinates of $\mathcal{F}_\lambda$ in Φ_X are given by the formula:

$$\lambda_i = \int_{\gamma_i} \mathcal{F}_\lambda d\mu, \tag{6}$$

where $\{\gamma_1, \ldots, \gamma_n\}$ is a basis in the group $H_1(X, Sing\ \mathcal{F}_\lambda; \mathbb{Z})$ and μ an invariant transversal measure on the leaves of foliation $\mathcal{F}_\lambda$. It follows from the formulas for the relative homology that:

$$n = 2g + |\lambda| - 1. \tag{7}$$

On the other hand, it is known that the number, k, of the linearly independent invariant measures of the foliation $\mathcal{F}_\lambda$ cannot exceed $\left[\frac{n}{2}\right]$ [13]. In view of formula (6), each independent measure $\mu_i, 1 \leq i \leq k$, defines a homomorphism $\mathbb{Z}^n \to \mathbb{R}$, whose kernel is a hyperplane in $\mathbb{R}^n$. Thus, one obtains a dimension group, G_λ, inside the abelian group $\mathbb{Z}^n$, which is bounded by the k hyperplanes corresponding to the measures $\mu_1, \ldots, \mu_k$.

Definition 1 *The dimension group G_λ is called a group associated to the simple geodesic γ.*

In the sequel, our main case will be $k = 1$ (i.e. the totally ordered dimension group G_λ). Note that the foliations $\mathcal{F}_\lambda$ with a unique invariant ergodic measure are generic in the space Φ_X [11]. In this generic case, the dimension group G_λ can be identified with a $\mathbb{Z}$-module $\mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_n$ in the real line $\mathbb{R}$.

4 Legendre laminations and their slopes on the modular surface

Let $\Gamma(N)$ be the principal congruence subgroup of level $N \geq 1$ and $X(N) = \mathbb{H}/\Gamma(N)$ the corresponding modular surface. The space of geodesic lamination on $X(N)$ is denoted by $\Lambda(X(N))$. It is known that $\Lambda(X(N))$ is a compact metric space endowed with the Chabauty topology [4].

Let $\alpha \in \mathbb{R} - \mathbb{Q}$ be a positive irrational, such that the regular continued fraction $\alpha = [a_0, a_1, a_2, \ldots]$ has a property that the matrices T_i:

$$\begin{aligned}
T_0 &= \begin{pmatrix} 0 & 1 \\ 1 & a_0 \end{pmatrix} \\
T_1 &= \begin{pmatrix} 0 & 1 \\ 1 & a_0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \\
T_2 &= \begin{pmatrix} 0 & 1 \\ 1 & a_0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_2 \end{pmatrix} \\
&\vdots
\end{aligned} \tag{8}$$

belong to the group $\Gamma(N)$. Let γ_i be an axis of the transformation T_i (see appendix). The geodesic lamination $\lambda \in \Lambda(X(N))$ is called *Legendre* if $\lambda = \lim_{i\to\infty} \gamma_i$ in the Chabauty topology.

Definition 2 *The positive irrational α is called a slope of the Legendre lamination λ on the modular surface $X(N)$.*

Recall that the geodesic laminations $\lambda, \lambda' \Lambda(X(N))$ are called *equivalent* whenever there exists a homeomorphism of the surface $X(N)$, which sends the (geodesic) leaves of λ into such of λ'.

Lemma 1 *The slopes α, α' of the equivalent Legendre laminations $\lambda, \lambda' \in \Lambda(X(N))$ are the equivalent irrationals $\alpha' = \alpha \bmod GL(2, \mathbb{Z})$.*

Proof. Let λ be the limit

$$\lambda = \lim_{n\to\infty} T_0 T_1 \ldots T_n, \tag{9}$$

and let $\varphi : X(N) \to X(N)$ be a homeomorphism sending λ to λ'. Let us find a limit formula for λ'. First, notice that there exists a bijection between T_i,

which occur in the limit of the Legendre laminations, and the set $\mathbb{Q}^+$ of positive rationals. (Indeed, T_i can be identified with the partial quotients of the regular continued fractions of irrational numbers.) Since φ acts on the set of closed geodesics of $X(N)$, which are the axes of T_i, the action of φ extends to the positive rationals. But any such an action has the form:

$$\varphi(r) = \frac{Ar+B}{Cr+D}, \quad r \in \mathbb{Q}^+, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(2, \mathbb{Z}^+). \tag{10}$$

One can use the euclidean algorithm to decompose $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ into the factors $T_0', \ldots, T_k'$, assuming that k is even. Therefore, the required limit formula for λ' has the form:

$$\lambda' = \lim_{n\to\infty} T_0' \ldots T_k' T_0 T_1 \ldots T_n \tag{11}$$

Since the regular continued fractions of the Legendre laminations λ and λ' have the same tails, we conclude that the slopes α, α' are the equivalent irrationals. $\square$

5 Proof of theorem 1

(i) Let $\lambda \in \Lambda(X(N))$ be a Legendre lamination and G_λ the corresponding dimension group. In view §4, we will write G_α, whenever λ has a slope α. Let us prove the following lemma.

Lemma 2 *The totally ordered dimension groups G_α and G_β are order-isomorphic if and only if*

$$\beta = \frac{a\alpha + b}{c\alpha + d}, \tag{12}$$

where a, b, c, d are integers such that $ad - bc = \pm 1$.

Proof. Since G_α is a dimension group with total order, the order is generated by a $\mathbb{Z}$-module $\mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_n$ in $\mathbb{R}$, where

$$\lambda_i = \int_{\gamma_i} \mathcal{F}_\alpha d\mu, \tag{13}$$

for a basis $\{\gamma_1, \ldots, \gamma_n\}$ in $H_1(X, Sing\ \mathcal{F}_\alpha; \mathbb{Z})$. On the other hand, for the topologically equivalent foliation $\mathcal{F}_\alpha, \mathcal{F}_\beta$ there exists a basis $\gamma_j' = \sum a_{ij}\gamma_i$, such that

$$\int_{\gamma_i} \mathcal{F}_\beta d\mu = \int_{\gamma_i'} \mathcal{F}_\alpha d\mu := \lambda_i', \tag{14}$$

where (a_{ij}) is an invertible integer matrix. Using the integration rules, one obtains:

$$\lambda_i' = \int_{\sum a_{ij}\gamma_i} \mathcal{F}_\alpha d\mu = \sum a_{ij}\lambda_i. \tag{15}$$

It is easy to see that the last equation implies that $\mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_n = \mathbb{Z}\lambda'_1 + \ldots + \mathbb{Z}\lambda'_n$, i.e. the dimension groups G_α, G_β are order-isomorphic. In view of lemma 1, it follows that $\beta = \alpha \ mod \ GL(2,\mathbb{Z})$. The inverse is proved similarly. $\square$

(ii) Let $\mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_n$ be a $\mathbb{Z}$-module in $\mathbb{R}$, corresponding to the totally ordered dimension group G_α. Consider the Jacobi-Perron continued fraction:

$$\begin{pmatrix} 1 \\ \theta \end{pmatrix} = \lim_{k\to\infty} \begin{pmatrix} 0 & 1 \\ I & b_0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix} \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix}, \tag{16}$$

where $\theta = (\lambda_2/\lambda_1, \ldots, \lambda_n/\lambda_1)$ and $\lambda_1 \neq 0$. It is known [2], that the fraction is convergent on a generic set $V \subset \mathbb{R}^n$ of vectors λ corresponding to the foliation $\mathcal{F}_\alpha$ with the unique ergodic measure. In particular, the totally ordered dimension groups are bijective with such foliations and therefore our fraction is always convergent.

One can now apply a result of Effros and Shen: the dimension group G_α is a unimodular dimension group given by the following limit of simplicial dimension groups:

$$\mathbb{Z}^n \xrightarrow{\begin{pmatrix} 0 & 1 \\ I & b_0 \end{pmatrix}} \mathbb{Z}^n \xrightarrow{\begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix}} \mathbb{Z}^n \ldots \tag{17}$$

see [7], Corollary 3.3. The item (ii) follows. $\square$

6 Appendix

The present section contains a brief account on the geodesic laminations, modular surfaces and the Jacobi-Perron continued fractions. The corresponding topics are covered in [4], [10] and [3], respectively.

6.1 Geodesics laminations

Let S be a surface of the constant negative curvature (a hyperbolic surface). By a *geodesic*, one understands a maximal arc on S consisting of the locally shortest sub-arcs (in the given metric on S). Each geodesic is the image of the open real interval I under a continuous map $I \to S$.

Lemma 3 (Topological classification of geodesics) *Let $p \in S$ be a point with an attached unit vector $t \in S^1$ on the surface S. Then :*

(a) for almost all points $t \in S^1$ the geodesic line through p in direction t is an immersion $I \to S$, i.e. a finite or infinite curve with the self-intersections;

(b) the complementary set $K \subset S^1$ has the cardinality of continuum and geodesic lines through p in the direction $t \in K$ are embedded curves of one of the three types:

(i) periodic

(ii) spiral

(iii) non-periodic whose closure is a perfect (Cantor) subset of S.

Proof. See [1] and [12]. □

The geodesics of type (b) are called *simple*, since they have no self-crossing or self-tangent points. Every geodesic $\gamma : I \to S$ of type (iii) is recurrent, i.e. for any $t_0 \in I$ and $\varepsilon > 0$ the ε-neighbourhood of $p(t_0)$ has infinitely many intersections with $\gamma(t)$ provided $t > N$, where $N = N(\varepsilon)$ is sufficiently large. The topological closure of recurrent geodesic on S contains continuum of the disjoint recurrent geodesics, and called a *geodesic lamination* λ. The intersection of λ with any closed curve on S is a Cantor set. The set $S-\lambda$ is called a *principal region* of λ. The principal region can have up to $4g-4$ connected components on the surface of genus $g \geq 2$ [4]. We denote by $|\lambda|$ the total number of such components.

A *foliation* $\mathcal{F}$ on a surface X is partition of X into a disjoint union of 1-dimensional and, possibly, a finite number of 0-dimensional leaves denoted by *Sing* $\mathcal{F}$. The immediate examples of foliations are orbits of the flows and trajectories of the quadratic differentials on X. The foliation $\mathcal{F}$ is called *measured* if it supports an invariant transversal measure on the leaves [8]. In other words:

(i) *Sing* $\mathcal{F}$ consists of the n-prong saddles, where $n \geq 3$;

(ii) each 1-leaf is everywhere dense in X.

The geodesic lamination λ can be obtained from $\mathcal{F}$ by a *blow-up* homotopy. Namely, a *separatrix* of $\mathcal{F}$ is a 1-leaf one of whose ends lie in *Sing* $\mathcal{F}$. The blow-up is a replacement of the separatrix by a narrow strip $[-\varepsilon, \varepsilon] \times \mathbb{R}$ using a homotopy surgery, which does not affect the nearby leaves. The complement of the blown-up separatrices consists of leaves of $\mathcal{F}$ that make up a perfect (Cantor) set on X. It is not hard to see that the above complement is homeomorphic to λ. Moreover, $|\lambda|$ is equal to the number of singular points of the foliation $\mathcal{F}$. Note that each measured foliations can be given by the orbits of a closed one-form ω on X, passing, if necessary, to a double cover of X [11].

6.2 Modular surfaces

Let $SL(2,\mathbb{Z})$ be the modular group. For every $N = 1, 2, 3, \ldots$, consider a normal subgroup $\Gamma(N) \subset \Gamma$ of finite index consisting of matrices of the form:

$$\begin{pmatrix} 1+Np & Nq \\ Nr & 1+Ns \end{pmatrix}, \tag{18}$$

where $p, q, r, s \in \mathbb{Z}$. The $\Gamma(N)$ is called a *principal congruence group* of level N. Let $\mathbb{H} = \{z = x + iy \in \mathbb{C} | y > 0\}$ be the Lobachevsky plane endowed with the hyperbolic metric $ds = |dz|/y$. The group $\Gamma(N)$ acts on $\mathbb{H}$ by the linear-fractional transformations $z \mapsto \frac{az+b}{cz+d}$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N)$. The orbit space $X(N) = \mathbb{H}/\Gamma(N)$ is an orbifold with a finite number of cusps corresponding to

the stabilizers of group $\Gamma(N)$. It follows from the construction that $X(N)$ is a Riemann surface, which is called a *modular surface*. The genus of $X(N)$ can be evaluated according to the classical formulas [10], p. 272.

Recall that an element $g \in SL(2,\mathbb{Z})$ is called (i) elliptic if $|tr\ (g)| < 2$, (ii) parabolic if $|tr\ (g)| = 2$, or (iii) hyperbolic if $|tr\ (g)| > 2$. The cases (i)-(iii) correspond to the transformations of $\mathbb{H}$ with: (i) one fixed point in $Int\ \mathbb{H}$, (ii) one fixed point in $\partial\mathbb{H}$, (iii) two fixed points in $\partial\mathbb{H}$. Let g be hyperbolic, and C the geodesic circle through the fixed points of g. It is known that $g(C) = C$ and g shifts points of C. The hyperbolic length of segment $[x, g(x)]$ does not depend on $x \in C$ and is calculated according to the formula $l([x, g(x)]) = 2\ arccosh\ \left(\frac{1}{2}|tr\ (g)|\right)$, where $tr\ (g)$ is the trace of matrix $g \in GL(2,\mathbb{Z})$. It is not hard to see that the segment $[x, g(x)]$ covers a periodic geodesic on $X(N)$, whose length coincides with $l([x, g(x)])$. The geodesic is called an *axis* of the transformation g.

6.3 Jacobi-Perron fractions

The Jacobi-Perron algorithm and connected (multidimensional) continued fraction generalizes the euclidean algorithm (regular continued fraction) of an irrational number. Namely, let $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_i \in \mathbb{R} - \mathbb{Q}$ and $\theta_{i-1} = \frac{\lambda_i}{\lambda_1}$, where $1 \leq i \leq n$. The continued fraction

$$\begin{pmatrix} 1 \\ \theta_1 \\ \vdots \\ \theta_{n-1} \end{pmatrix} = \lim_{k\to\infty} \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & b_1^{(1)} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & b_{n-1}^{(1)} \end{pmatrix} \cdots \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & b_1^{(k)} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & b_{n-1}^{(k)} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

where $b_i^{(j)} \in \mathbb{N} \cup \{0\}$, is called a *Jacobi-Perron fraction*. To recover the integers $b_i^{(k)}$ from the vector $(\theta_1, \ldots, \theta_{n-1})$, one has to repeatedly solve the following system of equations:

$$\left\{ \begin{array}{lll} \theta_1 & = b_1^{(1)} & + \frac{1}{\theta'_{n-1}} \\ \theta_2 & = b_2^{(1)} & + \frac{\theta'_1}{\theta'_{n-1}} \\ \vdots & & \\ \theta_{n-1} & = b_{n-1}^{(1)} & + \frac{\theta'_{n-2}}{\theta'_{n-1}}, \end{array} \right. \tag{19}$$

where $(\theta'_1, \ldots, \theta'_{n-1})$ is the next input vector. Thus, each vector $(\theta_1, \ldots, \theta_{n-1})$ gives rise to a formal Jacobi-Perron continued fraction. Whether the fraction is convergent or not, is yet to be determined.

Let us introduce the following notation. We let $A^{(0)} = \delta_{ij}$ (the Kronecker delta) and $A_i^{(k+n)} = \sum_{j=0}^{n-1} b_i^{(k)} A_i^{(\nu+j)}, \quad b_0^{(k)} = 1$, where $i = 0, \ldots, n-1$ and $k = 0, 1, \ldots, \infty$. The Jacobi-Perron continued fraction of vector $(\theta_1, \ldots, \theta_{n-1})$ is said to be *convergent*, if $\theta_i = \lim_{k\to\infty} \frac{A_i^{(k)}}{A_0^{(k)}}$ for all $i = 1, \ldots, n-1$. Unless $n = 2$, convergence of the Jacobi-Perron fractions is a delicate question. To

the best of our knowledge, there exists no intrinsic necessary and sufficient conditions for such a convergence. However, the Bauer criterion and the Masur-Veech theorem imply that the Jacobi-Perron fractions converge for the generic vectors $(\theta_1, \ldots, \theta_{n-1})$. Namely, let $\mathcal{F}$ be a measured foliation on the surface X of genus $g \geq 1$. Recall that the foliation $\mathcal{F}$ is called uniquely ergodic if every invariant measure of $\mathcal{F}$ is a multiple of the Lebesgue measure. By the Masur-Veech theorem, there exists a generic subset V in the space of all measured foliations, such that each $\mathcal{F} \in V$ is a uniquely ergodic measured foliation. We let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be the coordinate vector of the foliation $\mathcal{F}$. Then the following (Bauer's) criterion is true: the Jacobi-Perron continued fraction of λ converges if and only if $\lambda \in V \subset \mathbb{R}^n$ [2].

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